ISRAEL JOURNAL OF MATHEMATICS, Vol. 50, No. 3, 1985

AN EXTREMAL PROBLEM FOR DIRICHLET-FINITE HOLOMORPHIC FUNCTIONS ON RIEMANN SURFACES

BY

SHINJI YAMASHITA

Department of Mathematics, Tokyo Metropolitan University, Fukasawa, Setagaya, Tokyo 158, Japan

ABSTRACT

If f is a nonconstant holomorphic function with finite Dirichlet integral $D(f)$ on a Riemann surface R, then $|f|^2$ has the least harmonic majorant f_2 on R. We show $\Sigma f_2(a) \leqq \pi^{-1}D(f)$, where a runs over all the roots of $f=0$ on R. The equality holds if and only if f is of type $\mathcal{B}\ell_1$ from R onto a disk of center 0. A consideration is proposed for the non-Euclidean case.

1. Introduction

Let R be a Riemann surface having the Green functions $g_R(z,w)$ with poles $w \in R$. Each function f holomorphic and Dirichlet-finite on R,

$$\int\int_R |f'(z)|^2 dxdy < \infty, \qquad z = x+iy,$$

is then known to be in the Hardy class $H^2(R)$ in the sense that the subharmonic function $|f|^2$ is majorized by a harmonic function h on R, that is, $|f|^2 \leqq h$ on R. Therefore, the least harmonic majorant f_2 of $|f|^2$, the smallest among all the harmonic functions majorizing $|f|^2$ on R, exists. Our main result is

THEOREM 1. *Let f be a function nonconstant, holomorphic, and Dirichlet-finite on R. Then*

$$(1.1) \qquad \sum_{f(a)=0} f_2(a) \leqq \pi^{-1} \int\int_R |f'(z)|^2 dxdy,$$

where a in the left summation runs over all the zeros of f, the multiplicities being

Received May 10, 1984

counted. The equality in (1.1) *holds if and only if f is of type $\mathcal{B}\ell_1$ from R onto an open disk $\{|w|<\rho\}$, $\rho>0$, in the complex plane.*

The left-hand side of (1.1) is zero if f has no zero. A nonconstant analytic map f from R into another Riemann surface S, also having Green's functions $g_S(z,w)$, is said to be of type $\mathcal{B}\ell_1$ in the sense of Heins [2, p. 447] if for each $w \in S$, the equality holds in R:

$$g_S(f(z),w)=\sum_{f(a)=w} g_R(z,a), \qquad z\in R, \tag{1.2}$$

where a runs over all the w-points of f, the multiplicities being counted.

If f is of type $\mathcal{B}\ell_1$ from R onto $\{|w|<\rho\}$ and if f is Dirichlet-finite, then for each w, $|w|<\rho$, the number of the roots of $f=w$ on R is finite and the same by [2, Theorem 21.2, p. 470].

In the specified case $R=S=\Delta\equiv\{|w|<1\}$, and f is Dirichlet-finite, we readily observe that f is of type $\mathcal{B}\ell_1$ if and only if f is a finite Blaschke product. For the proof we just make use of Frostman's theorem [1, Theorem 2.14, p. 35]. Thus, in the case $R=\Delta$, the equality in (1.1) holds if and only if f is a finite Blaschke product multiplied by a positive constant.

2. An identity

The inequality (1.1) is an immediate consequence of the identity (2.1) in the following

THEOREM 2. *For f in the assumption of Theorem* 1 *we have*

$$\sum_{f(a)=0} f_2(a)=\pi^{-1}\iint_R |f'(z)|^2dxdy-4\pi^{-1}\iint_R |(\partial/\partial z)\{|f(z)|e^{g(z)}\}|^2dxdy, \tag{2.1}$$

where

$$g(z)=\sum_{f(a)=0} g_R(z,a), \qquad \partial/\partial z=\tfrac{1}{2}(\partial/\partial x-i\partial/\partial y).$$

PROOF. Let D be a subdomain of R, whose closure $D\cup\partial D$ is compact and whose boundary ∂D consists of a finite number of mutually disjoint, analytic Jordan curves. Let a_k $(k\geqq 1)$ be all the distinct zeros of f in D, and let n_k be their orders. We shall prove (2.1) for $R=D$ first, namely,

$$\sum_k n_k f_{2,D}(a_k)=\pi^{-1}\iint_D |f'(z)|^2dxdy-4\pi^{-1}\iint_D |(\partial/\partial z)\{|f(z)|e^{g_D(z)}\}|^2dxdy, \tag{2.2}$$

where $f_{2,D}$ is the least harmonic majorant of $|f|^2$ in D, and

$$g_D(z) = \sum_k n_k g_D(z, a_k),$$

$g_D(z, a_k)$ being the Green function of D with pole at a_k. Since there exists a sequence $\{D_n\}_{n=1}^{\infty}$ of the domains of the described type on R exhausting R in the sense that $D_n \cup \partial D_n \subset D_{n+1}$ $(n \geqq 1)$, $R = \bigcup_n D_n$, we obtain (2.1) on letting $D = D_n$ in (2.2) and then letting $n \to \infty$. Since the Green function $g_{D_n}(z, w)$ converges to $g_R(z, w)$ locally and uniformly in $R \setminus \{w\}$, the limiting process is possible.

For the proof of (2.2) we choose $\gamma_k = \{|z - a_k| \leqq \varepsilon\}$ in each parameter disk of center a_k, where $\varepsilon > 0$ is common for all k, such that $D = D \setminus \bigcup_k \gamma_k$ is again a domain of the described type. By the Green formula,

$$\begin{aligned}\iint_{D_\varepsilon} \Delta(|f|^2)dxdy - \iint_{D_\varepsilon} \Delta(|f|^2 e^{2g_D})dxdy &= \iint_{D_\varepsilon} \Delta\{(1 - e^{2g_D})|f|^2\}dxdy \\ &= \int_{\partial D_\varepsilon} (\partial/\partial\nu)\{(1 - e^{2g_D})|f|^2\}ds, \end{aligned} \tag{2.3}$$

where $\partial/\partial\nu$ means the differentiation in the direction of the outward-pointing normal and ds is the element of the arc length. The integrand in the last line of (2.3) is divided as

on ∂D:

$$-2|f|^2 \partial g_D/\partial\nu = -2\sum_k n_k |f|^2 \partial g_D(z, a_k)/\partial\nu;$$

on $\partial\gamma_k$:

$$-[(\partial/\partial r)|f(r\zeta + a_k)|^2]_{r=\varepsilon} + [(\partial/\partial r)|f(r\zeta + a_k)|^2 \exp\{2g_D(r\zeta + a_k)\}]_{r=\varepsilon},$$

where $|\zeta| = 1$. Since $\Delta(|f|^2) = 4|f'|^2$ in D_ε, and

$$f_{2,D}(a_k) = -(2\pi)^{-1}\int_{\partial D} |f|^2 \partial g_D(z, a_k)/\partial\nu ds, \qquad k \geqq 1;$$

$$\Delta(|f|^2 e^{2g_D}) = 16|(\partial/\partial z)(|f|e^{g_D})|^2 \qquad \text{in } D_\varepsilon,$$

we obtain, on letting $\varepsilon \to 0$ in (2.3), the identity

$$4\iint_D |f'(z)|^2 dxdy - 16\iint_D |(\partial/\partial z)\{|f(z)|e^{g_D(z)}\}|^2 dxdy = 4\pi\sum_k n_k f_{2,D}(a_k),$$

whence (2.2).

3. Proof of Theorem 1

It remains to prove when the equality in (1.1) holds. To prove the "if" part we may assume without loss of generality that $\rho = 1$. It follows from

$$-\log |f(z)| = g_\Delta(f(z),0) = g(z) \qquad \text{in } R,$$

that $|f| e^g = 1$ in (2.1), whence the equality in (1.1) holds.

To prove the "only if" part we suppose that $|f| e^g$ is a positive constant, which we may assume to be 1, on R in (2.1). Since $|f| = e^{-g} < 1$ on R, $f(R) \subset \Delta$. Therefore,

$$-\log |w| + H(w) \equiv g_{f(R)}(w,0) \leqq g_\Delta(w,0) = -\log |w \qquad \text{in } f(R),$$

whence $H \leqq 0$. By [2, (1.1), p. 440],

$$g_{f(R)}(f(z),0) = g(z) + u(z) = -\log |f(z)| + u(z),$$

where $u \geqq 0$ is harmonic on R. Since

$$0 \leqq u(z) \equiv H(f(z)) \leqq 0 \qquad \text{on } R,$$

it then follows that $u \equiv H \equiv 0$.

For the shape of $f(R)$, we note that $g_{f(R)}(w,0) = -\log |w|$ for $w \in f(R)$. Hence $\Delta \cap \partial f(R) = \varnothing$ on considering the boundary values of $g_{f(R)}(w,0)$, so that $f(R) = \Delta$.

Since the quasibounded part of the Parreau decomposition of u is constant zero, $f : R \to \Delta$ is of type $\mathcal{B}\ell$ by [2, Theorem 4.1, p. 446]. Since f is Dirichlet-finite, the valence function ν_f [2, (8.5), p. 455] is finite for a.e. point of Δ. By the first half of [2, Theorem 21.2, p. 470], $\sup \nu_f < \infty$. In particular, the last term in

$$-\log |f(z)| = g(z) = \sum_{f(a)=0} g_R(z,a) \tag{3.1}$$

is a finite sum.

Let $\pi : \Delta \to R$ be a universal covering map. By Myberg's theorem [3, Theorem XI.13, p. 522],

$$g_R(\pi(w), a) = \sum_{\pi(b)=a} g_\Delta(w,b), \qquad w \in \Delta,$$

together with

$$\sum_{\pi(b)=a} (1 - |b|) < \infty. \tag{3.2}$$

Combining (3.1) and (3.2) we find

$$-\log |(f \circ \pi)(w)| = \sum_{f(a)=0} \sum_{\pi(b)=a} g_\Delta(w, b), \qquad w \in \Delta,$$

with

$$\sum_{f(a)=0} \sum_{\pi(b)=a} (1 - |b|) < \infty.$$

By Littlewood's theorem [3, Theorem IV.33, p. 170],

$$\lim_{r \to 1-0} \int_{\partial\Delta} \log |(f \circ \pi)(r\zeta)| \, |d\zeta| = 0.$$

By Riesz's theorem [1, Theorem 2.12, p. 32], we observe that $f \circ \pi$ is a Blaschke product. Furthermore, the Riemannian image of Δ by $f \circ \pi$, which coincides with that of R by f, has finite area, so that $f \circ \pi$ must be a finite Blaschke product. Since $f \circ \pi$ is of type $\mathcal{B}\ell_1$, it follows from [2, Corollary, p. 472] that f itself must be of type $\mathcal{B}\ell_1$ and our Theorem 1 is established.

4. Non-Euclidean area

For f nonconstant, holomorphic, and bounded, $|f| < 1$, on R, the non-Euclidean area of the Riemannian image of R by f is

$$\iint_R |f'(z)|^2/(1 - |f(z)|^2)^2 dxdy. \tag{4.1}$$

What is a counterpart of (2.1) in this case? We prove

THEOREM 3. *For f described above with finite integral* (4.1), *we have*

$$\begin{aligned}\sum_{f(a)=0} f_\lambda(a) = \pi^{-1} \iint_R |f'(z)|^2/(1 - |f(z)|^2)^2 dxdy \\ - \pi^{-1} \iint_R e^{2g}\lambda(f)[4^{-1}\Delta(\log \lambda(f)) + |(\partial/\partial z)(2g + \log \lambda(f))|^2] dxdy,\end{aligned} \tag{4.2}$$

where $\lambda(f) = -\log(1 - |f|^2)$, f_λ is the least harmonic majorant of $\lambda(f)$ in R, and g is the same as in Theorem 2.

Note that $\Delta \log \lambda(f) \geqq 0$, and $4^{-1}\Delta[e^{2g}\lambda(f)]$ is the same as the integrand in the second integral in (4.2). The proof is in spirit the same as that of Theorem 2. As an immediate consequence of (4.2) we obtain the strict inequality:

$$\sum_{f(a)=0} f_\lambda(a) < \pi^{-1} \iint_R |f'(z)|^2/(1 - |f(z)|^2) dxdy,$$

a counterpart of (2.1).

Added in proof. The proof of the "only if" part in the last sentence in Theorem 1 given in Section 3 is not true for general R. We suppose that R is regular [2, p. 455], and follow the proof up to the inequality $\sup \nu_f < \infty$ just before (3.1) (we omit the remaining part in Section 3). It then follows from [2, Theorem 8.1, p. 455] that for each $q \in f(R) = \Delta$, the singular harmonic function

$$g_\Delta(f(z), q) - \sum_{f(a)=q} g_R(z, a), \qquad z \in R,$$

must be zero, whence f is of type $\mathcal{B}\ell_1$.

References

1. E. F. Collingwood and A. J. Lohwater, *The Theory of Cluster Sets*, Cambridge University Press, Cambridge, 1966.

2. M. Heins, *On the Lindelöf principle*, Ann. Math. **61** (1955), 440–473.

3. M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen, Tokyo, 1959.